Lattie Sieving 1

Lattice Sieving Algorithms and Their Implementation

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Section 1

Introduction
Introduction

In this talk, we talk about sieving algorithm for solving (approximation) SVP on lattice.
Outline

• Introduction
  This it is. The outline, background.

• Basics
  Review the basics of lattice, GSO, and basis reduction.

• The Sieving Algorithms
  The core of sieving algorithms, the intuition, and its evolution

• The Sieving Tricks
  Algorithms beyond sievers that accelerates sieve

• G6K Implementation
  The overview of sieving library G6K that implements all above

• Future Plan
  Our plan on sieving research.
SVP Solvers

SVP is a **NP**-hard lattice problem that can be solved by following exponential complex algorithms:

- **Enumeration** (super exponential time, polynomial memory)
- **Sieving** (exponential time, exponential memory)
- **Basis Reduction** (often heuristic time, often only solves uSVP)

Enumeration was once faster than sieving in practice, but sieving is now faster in both theoretical and practical.
Main Sieving Breakthroughs

- [NV08] shows that sieving is practical, and solved for dim 50
- [LM18] The totally heuristic progressive sieve speeds up several algorithms for more than 20x!
- [ADH\textsuperscript{+}19] implements multithread sieving with several tricks that solves 1.05-SVP for about 150 dims.
- [DSvW21] implements sieving on GPUs that solves 1.05-SVP for 180 dims, spending 51.6 days with 4 RTX 2080Ti and 1.5TB RAM.
Section 2

Basics
Lattices, Basis, Volume

Let \( \mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m \in \mathbb{R}^n \) and be linear independent. Let \( \mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_m] \), then we write

\[
\Lambda = \mathcal{L}(\mathbf{B}) = \left\{ \sum_{i \in [m]} x_i \mathbf{b}_i \mid x_i \in \mathbb{Z} \right\}
\]

We call \( \mathbf{B} \) a \textbf{basis} of lattice \( \Lambda \).
We define the volume of \( \mathcal{L}(\mathbf{B}) \) to be \( |\det \mathbf{B}| \).
Shortest Vector Problem

**Shortest Vector Problem (SVP)**

Given a lattice basis $B \in \mathbb{Z}^{m \times n}$, find a nonzero lattice vector $x \in (B)$ s.t. $\|x\| \leq \|y\|$ for any other $y \in \mathcal{L}(B)$.

Denote the vector length of SVP in Lattice $\mathcal{L}$ by $\lambda_1(\mathcal{L})$. We abuse notion $\lambda_1(B)$ for $\lambda_1(\mathcal{L}(B))$.

**$\gamma$-Approximate Shortest Vector Problem (SVP$_\gamma$)**

Given a lattice basis $B \in \mathbb{Z}^{m \times n}$, find a nonzero lattice vector $x \in (B)$ s.t. $\|x\| \leq \gamma(n) \cdot \lambda_1(B)$ for any other $y \in \mathcal{L}(B)$.

We mainly focus on solving these 2 problems.
Gram-Schmidt Orthogonalization

For a lattice basis $B = [b_0, \cdots, b_{n-1}] \in \mathbb{R}^{m \times n}$, the Gram-Schmidt orthogonal basis of $B$ can be compute by

$$
\begin{align*}
    b_0^* &= b_0 \\
    b_1^* &= b_1 - \mu_{1,0} b_0^* \\
    b_{n-1}^* &= b_{n-1} - \sum_{j=0}^{n-2} \mu_{n-1,j}(b_j^*)
\end{align*}
$$

where $\mu_{i,j} = \frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle}$. We say $B^* = [b_0^*, \cdots, b_{n-1}^*]$ the Gram-Schmidt Orthogonal Form of $B$. 

Other Notations

Let \( v \in \mathcal{L}(B) \), define

\[
\pi_\ell(v) := v - \sum_{i=0}^{\ell-1} \frac{\langle b_i^*, v \rangle}{\langle b_i^*, b_i^* \rangle} b_i^*
\]

It projects \( v \) to the perpendicular space of \( \text{span}(b_0, \cdots, b_{\ell-1}) \)

We denote

\[
\pi_\ell(B) := [\pi_\ell(b_0), \cdots, \pi_\ell(b_{n-1})]
\]

We abuse \( \pi_\ell(\mathcal{L}) \), where \( \mathcal{L} = \mathcal{L}(B) \) for \( \mathcal{L}(\pi_\ell(B)) \).

Denote

\[
B_{[\ell,r]} := [\pi_\ell(b_\ell), \cdots, \pi_\ell(b_r)]
\]

And we denote \( B_{[\ell,n]} \) by \( B_\ell \).
Other Notations

$B_n(R)$: the ball with radius $R$ and center $0$ in $\mathbb{R}^n$. 
Basis Reduction

We have following reduced conditions, $\mathbf{B}$ is

- size-reduced, if $|\mu_{i,j}| \leq 1/2$ for all $0 \leq j < i < n$.
- $\delta$-LLL-reduced, if it is sized reduced and $(\delta - \mu_{i+1,i})\|\mathbf{b}_i^*\|^2 \leq \|\mathbf{b}_{i+1}^*\|^2$. ($1/4 < \delta \leq 1$)
- HKZ-reduced, if $\mathbf{b}_i^*$ is a shortest vector of $\pi_i(\mathcal{L})$ for all $1 \leq i < n$.
- $\beta$-BKZ-reduced, if sublattices $\mathcal{L}(\mathbf{B}_{[\ell,\ell+\beta]})$ is HKZ-reduced for all $0 \leq \beta < n - \ell$. 
The Sievers
From Enumeration to Sieving

Enumerations solves SVP.

Let $S \subseteq \mathbb{R}^n$ be a subset containing $\lambda_1(\mathcal{L})$, retrieving all the vectors in $\mathcal{L} \cap S$ would give a vector of length $\lambda_1(\mathcal{L})$.

Sadly, the best such $S$ contains $2^{O(n \log n)}$ vectors of $\mathcal{L}$ which leads to $2^{O(n \log n)}$ time complexity.

Sieving as Randomized Enumeration

Sieving algorithms choose smaller $S$, that $S \cap L$ is hard to be enumerated. However, SVP can be solved by random sampling. Choose $R = O(\lambda_1(\mathcal{L}))$ leads to $|\mathcal{L} \cap B_n(R)| = 2^{O(n)}$. 
AKS Sieving

AKS sieving is proposed by Ajtai, Kumar and Sivakumar in [AKS01]. If we know how to uniformly sampling from such $\mathcal{L} \cap B_n(R)$, we are done. However, we don’t.

Sampling from larger $\mathcal{L} \cap B_n(R_0)$ where $R_0 = 2^{O(n)} \lambda_1(\mathcal{L})$ is easy for us, so we sample from $\mathcal{L} \cap B_n(R_0)$ instead.

Algorithm 2 Initial sampling

| Input: A basis $B = [b_1, \ldots, b_n]$ of a lattice $L$, and a real $\xi > 0$. |
| Output: A pair $(v, y) \in L \times \mathbb{R}^n$ such that $\|y\| \leq n \max_i \|b_i\|$ and $y - v$ is uniformly distributed in $B_n(\xi)$. |
| 1: $x \leftarrow_{\text{random}} B_n(\xi)$ |
| 2: $v \leftarrow \text{ApproxCVP}(-x, B)$ where ApproxCVP is Babai’s rounding algorithm [6]. |
| 3: $y \leftarrow v + x$ |
| 4: return $(v, y)$ |

Where ApproxCVP is Babai’s rounding algorithm.
To obtain $L \cap B_n(R)$ for $R = O(\lambda_1(L))$, we “sieve” the database $L$. Namely, use shorter vector to substitute long vectors in $L$.

Algorithm 3 The sieve with perturbations

**Input:** A set $S = \{(v_i, y_i), \ i \in I\} \subseteq L \times B_n(R)$ and a triplet $(\gamma, R, \xi)$ such that $\forall i \in I, \|y_i - v_i\| \leq \xi$.

**Output:** A set $S' = \{(v'_i, y'_i), \ i \in I'\} \subseteq L \times B_n(\gamma R + \xi)$ such that $\forall i \in I', \|y'_i - v'_i\| \leq \xi$.

1: $C \leftarrow \emptyset$
2: for $i \in I$ do
3: if $\exists c \in C \|y_i - y_c\| \leq \gamma R$ then
4: $S' \leftarrow S' \cup \{(v_i - v_c, y_i - v_c)\}$
5: else
6: $C \leftarrow C \cup \{i\}$
7: end if
8: end for
9: return $S'$
AKS Sieving

In every round of sieving, we wish to decrease the database vector length by a factor of \( \gamma \) by paying \( |C| \) vectors.

Some comments:

- We sieve the perturbated lattice vectors in sieving procedure, while recording their corresponding lattice vectors. This is necessary for correctness proof (which says we often obtain a nonzero shortest vector eventually).
- The sieving procedure works if we can bound \( \lambda_1 \) with some constants \( \lambda \) s.t. \( \lambda < \lambda_1 < 1.01\lambda \). We have to try polynomial numbers of \( \lambda \)'s if we don’t know such range.

NV sieving introduce heuristics to overcome these performance drawbacks.

- Assume the vectors after sieving distributes uniformly.
- Introduce Gaussian heuristic to estimate the \( \lambda_1 \).
Gaussian Heuristic

Definition (Gaussian heuristic)
Let $V$ be a measurable subset of $\mathbb{R}^n$, then $|V \cap \mathcal{L}|$ is approximately equal to $\frac{\text{vol}(V)}{\text{vol}(\mathcal{L})}$.

We have
\[
\text{vol}(B_n(R)) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} R^n \approx \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} R^n
\]

By letting it equal to $\text{vol}(\mathcal{L})$ and solve for $R$, we obtain an approximation of $\lambda_1$,

\[
\text{gh}(\mathcal{L}) = \sqrt{\frac{n}{2\pi e}} \cdot \text{vol}(\mathcal{L})^{1/n}
\]

We denote the gaussian heuristic of $n$-dimensional lattice with volume 1 by $\text{gh}(n)$. 
Gaussian Heuristic

**Definition**

Gaussian heuristic: Let $V$ be a measurable subset of $\mathbb{R}^n$, then $|V \cap \mathcal{L}|$ is approximately euqal to $\text{vol}(V)/\text{vol}(\mathcal{L})$.

We have

$$\text{vol}(B_n(R)) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} R^n \approx \frac{1}{\sqrt{n\pi}} \left(\frac{2\pi e}{n}\right)^{n/2} R^n$$

By letting it equal to $\text{vol}(\mathcal{L})$ and solve for $R$, we obtain an approximation of $\lambda_1$,

$$\text{gh}(n) = \sqrt{n/2\pi e} \cdot \text{vol}(\mathcal{L})^{1/n}$$

The $\text{gh}(n)$ works well for predicting the $\lambda_1$ of random lattice with dim larger than 50.
NV Sieving

NV sieve proposed by Nguyen and Vidick in [NV08] makes additional following assumption to simplify AKS sieve.

**Definition (Heuristic)**

Any stage after AKS sieve, the vectors in the database distributes uniformly in

\[ C_n(\gamma, R) = \{ x \in \mathbb{R}^n : \gamma R \leq \|x\| \leq R \} \]

This says we have negligible probability to lose vectors in database by collision until we achieve approximately \( \lambda_1 \)-length vectors.

**NV sieve doesn’t promise an exact-SVP solution, but we often obtain such one in practice.**
NV Sieving choose a database of size $\sqrt{4/3^n}$, due to next lemma.

**Lemma**

Let $n \in \mathbb{N}$ and $2/3 < \gamma < 1$. Define $c_H = 1/ \left( \gamma \sqrt{1 - \gamma^2/4} \right)$ and $N_C = c_H^n \left[ 3\sqrt{2\pi}(n+1)^{3/2} \right]$. Let $N$ be an integer, and $S$ a subset of $C_n(\gamma, R)$ of cardinality $N$ whose points are picked independently at random with uniform distribution.

- If $N_C < N < 2^n$, then for any subset $C \subseteq S$ of size at least $N_C$ whose points are picked independently at random with uniform distribution, with overwhelming probability, for all $v \in S$, there exists a $c \in C$ such that $\|v - c\| \leq \gamma$.

- If $N < 4\sqrt{\pi/2n}\sqrt{4/3^n}$, the expected number of points in $S$ that are at distance at least $\gamma$ from all the other points in $S$ is at least $(1 - 1/n)N$. 
NV Sieving

From last lemma, we have the following:

- Choose gamma close to 1 allows us to have database of size $N = \sqrt{4/3 + \epsilon^n}$, which gives running time of $O((4/3)^n)$.

- Database with size less than $N = \Omega(n^{-1/2}\sqrt{4/3}^n)$ is impossible since it loses $(1 - 1/n)N$ points in every round of sieving.

This forces us to use database with size $O(\sqrt{4/3}^n) \approx O(2^{0.2075})$. 
NV Sieve+

We do not decrease the database vector base size with a factor of $\gamma$. We continuously consider pairs $\mathbf{v}_1, \mathbf{v}_2$ in the database. The NV Sieve+ algorithm works as follows:

- **Sieve:** Replace the longest vector in the database with $|\mathbf{v}_1 \pm \mathbf{v}_2|$ if $|\mathbf{v}_1 \pm \mathbf{v}_2| < \max\{|\mathbf{v}| : \mathbf{v} \in \text{db}\}$.
- **Stop:** Stop when database is filled with $\sigma \sqrt{4/3^n}$ vectors in $B_n(\sqrt{4/3\gamma h(L)})$.
- **Output:** Output the shortest $\mathbf{v}_1 \pm \mathbf{v}_2$ for $\mathbf{v}_1, \mathbf{v}_2 \in \text{db}$.

$\sigma$ is called the saturation ratio.
We obtain our general sieve algorithm from NV sieve+.

**Algorithm 1:** Lattice sieving algorithm.

**Input:** A basis $\mathbf{B}$ of a lattice $\mathcal{L}$, list size $N$ and a saturation radius $R$.

**Output:** A list $L$ of short vectors saturating the ball of radius $R$.

1. Sample a list $L \subset \mathcal{L}$ of size $N$.
2. while $L$ does not saturate the ball of radius $R$ do
3.     for every pair $\mathbf{v}, \mathbf{w} \in L$ do
4.         if $\mathbf{v} - \mathbf{w} \notin L$ and $\|\mathbf{v} - \mathbf{w}\| < \max_{\mathbf{u} \in L} \|\mathbf{u}\|$ then
5.             Replace a longest element of $L$ by $\mathbf{v} - \mathbf{w}$.
6. return $L$
Gaussian Sieve

Gaussian sieve is proposed by Micciancio and Voulgaris in [MV10]. It use Gaussian reduction in sieve.

Gauss divides the database in two parts:

- **Queue**: Always sorted, any \( v_1, v_2 \in \text{Queue} \) are size-reduced.
- **List**: The rest part of db.

Every pairs \( v_1, v_2 \in \text{Queue} \) is Gaussian reduced.

The operations, asymptotic running time, database size is similar to NV Sieve+, but works better in practice.
Bucketing

Checking pairs in db exhaustively is time-consuming. Thus we put vectors in db into buckets, and only check pairs in the same bucket. Notice that a vector can be put into multiple buckets.

- **bgj1**: Choose uniformly $d \in \mathbb{R}^n$ s.t. $\|d\| = 1$. An vector $v \in \text{db}$ is put into $bd$ if $|\langle v, d \rangle| > \alpha \cdot |v|$ for constant $\alpha$. Time complexity $2^{0.349n+o(n)}$, space complexity $2^{0.2075n+o(n)}$.

- **bdgl**: Split the dimension $n$ into dimension $n_1, \cdots, n_k$ s.t. $n_1 + \cdots + n_k = n$. Randomly choose centers $C_k$ for each subdimension. Then the global bucket centers are $C = C_1 \times \pm C_2 \times \cdots \times \pm C_k$. For a vector $v$, it is firstly bucketed into closests bucket centers and then put into global centers.

$bdgl$ is faster asymptotically, while $bgj1$ is practically faster. Both are implemented in G6K-GPU.
Tuple Sieve

Consider $t$-tuples $\mathbf{v}_1, \cdots, \mathbf{v}_t \in \text{db}$ and replace longests with $\mathbf{v}_1 \pm \mathbf{v}_2 \pm \cdots \pm \mathbf{v}_t$ if these are shorter.

• More time-consuming but saves memory due to the good performance for smaller database.

• We set $t = 3$, i.e., use triple sieve, in practice.

• Also supports bucketing.
  For bucket with center $\mathbf{x}$ being a lattice vector, we test for $\mathbf{v}_1 \pm \mathbf{v}_2 \pm \mathbf{x}$ for each pair $\mathbf{v}_1, \mathbf{v}_2$ in the bucket. Time complexity $2^{O(0.396n+o(n))}$, space complexity $2^{O(0.1788n+O(n))}$. 
Section 4

The Sieving Tricks
Sublattice Vector Lifting

Let $w$ be a vector from sublattice $B_\ell$, how can we recover $x$ s.t. $\pi_\ell(x) = w$? This problem asks us to “lift” $w$ back to $w$ by undoing the projections.

There are uncountable legal lifts of $w$.

An arbitrary lift might be meaningless, we are committed to find short lifts.
Babai’s Lift

Find the shortest lift for 1-dim is easy:

• Undo all the projections.
• Find the shortest lift by rounding.

We lift \( \mathbf{w} = B_\ell \mathbf{v} \) to \( \mathbf{w}' = B_{\ell-1} \mathbf{v}' \). Obviously, \( \mathbf{w}_0 = B_{\ell-1}(0, \mathbf{v}) \) is a lift, but not the best. By lift \( \mathbf{w} \) tp \( \mathbf{w}_0 \), we have add

\[
\sum_{i=\ell}^{n-1} \mu_{i,\ell-1} v_i b_{\ell-1}^*
\]

to \( \mathbf{w} \). To add at least \( b_{\ell-1}^* \) as possible, we set \( v_{\ell-1} = \lfloor c \rfloor \) in \( \mathbf{v}' \). This allows us to add less than \( 1/2 \ b_{\ell-1}^* \) in \( \mathbf{w}' \) to \( \mathbf{w} \) and is the best.
Babai’s Lift

To lift for multiple dims, we repeat Babai’s Lifting greedy. This seldom gives the best lift, but gives an well enough one.

Achieving the best lift for \( d \)-dim is hard. Which is equal to solve a CVP problem of dimensions \( d \).

We next figure out when Babai’s lift gives a best lift (for dim \( d \)). With this kind of lift, we solve SVP on \( \mathcal{L}_d \) by sieving and lift its SVP to the SVP of \( \mathcal{L} \). Then we have \( d \) dimensions for free.
We analysis dim for free with NV sieve, but it can be used else where.
**Dim for Free**

Let

\[ L := \text{NvSieve} = \{ x \in \mathcal{L}_d \setminus \{0\} : \|x\| \leq \sqrt{\frac{4}{3} \cdot \text{gh}(\mathcal{L}_d)} \} \]

Let \( s \) be the SVP of \( \mathcal{L} \). To have \( d \)-dims for free, we wish to have \( \pi_d(s) \in L \). Which gives

\[ \text{gh}(\mathcal{L}) \leq \sqrt{\frac{4}{3} \cdot \text{gh}(\mathcal{L}_d)} \]  

(1)

To lift \( \pi_d(s) \) to \( s \) by Babai’s lifting, a sufficient condition is \( |\langle b_i^*, s \rangle| \leq \|b_i^*\|^2 \) for all \( i < d \). This achieves by

\[ \text{gh}(\mathcal{L}) \leq \frac{1}{2} \min_{i<d} \|b_i^*\| \]

(2)

which would not be a serious issue for small \( d \) in practice.
To justify equation (1), we resort to BKZ-reduction.

**Definition**

Geometric Series Assumption Let $\mathbf{B}$ be a $b$-BKZ reduced basis of a lattice of volume 1, the geometric assumption states that:

$$\|\mathbf{b}_i^*\| = \alpha_b \left(\frac{n-1}{2^i}\right)$$

where $\alpha_b = gh(b)^2/b$.

For $b$-BKZ basis $\mathbf{B}$ of $\mathcal{L}$, we have

$$\text{vol}(\mathcal{L}_d) = \prod_{i=d}^{n-1} \|\mathbf{b}_i^*\| = \alpha_b^{d(d-n)/2}$$

which can be approximately rewrite as

$$d \ln \alpha_b \leq \ln\left(\frac{4}{3}\right) + \ln\left(1 - \frac{d}{n}\right)$$

(3)
Dim for Free

Take $b = n/2$ s.t. the time complexity of doing BKZ-reduction is negligible compared to such of sieving, we can achieve equation (3) with $d = \Theta(n/\ln n)$.

Lift On the Fly
To obtain more concrete dim for free, we lift every vector after in the sieving procedure, right after is has been obtained (even if it is not small enough for current sublattice basis). This is called “lift on the fly”.

The concrete dim4free dimension number has to be obtain by doing experiment. In this case, BKZ-reduction is not necessary.
Progressive Sieve

Instead of work with dimension $n$ directly, progressive sieve [LM18] increase dimension number gradually.

Progressive starts with sublattice of small dimension and its database, and iterates for the following until we achieve the goal dimension.

1. Do sieving to achieve good database under such dimension.
2. Increase sublattice dimension a little. Lift all the basis in db to new current sublattice vectors.
3. Sample new current sublattice vectors to db.

In implementation, we starts from $L_{[n,n]}$, and iterate for $L_{[n-1,n]}$, $L_{[n-2,n]}$, $\cdots$, $L_{[1,n]}$. Progressive sieve is totally heuristic, but works well in practice.
Flexible Insertion

During sublattice sieving, we insert good vector in the db back to basis $B$ to improve it. The ideal comes from the sampling reduction algorithms. We choose the order of insert by scoring.

**Scoring.** We keep insertion candidates $c_i$ (can be $\perp$ sometimes) for each position, we scoring them by

$$\varsigma(i) = \begin{cases} 0, & \text{if } c_i = \perp \\ \theta^{-i} \cdot \| b_i^* \|^2 / \| c_i \|^2, & \text{otherwise} \end{cases}$$

and insert at position $i$ with highest score first.

After insertion, the current sublattice basis will changed. Thus we have to recompute the vectors db.
Fast Pair/Triple Rejection - POPCNT/SimHash

Population count allows us to reject bad pairs and triples fast. We explain for pairs as an example. Since

\[ \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2\langle v_1, v_2 \rangle \]

We tend to reject \( v_1, v_2 \) with the same sign on the same component.

In implementation, we sample 256 random vectors \( r_1, \cdots, r_{256} \) vectors, and store the signs of \( \langle v, r_i \rangle \) for each \( v \in \text{db} \). This is the SimHash of \( v \).

The counting of positives in \( \text{SimHash}(v_1) \oplus \text{SimHash}(v_2) \) can be done by an efficient modern CPU instruction POPCNT, which is provided by std::popcount in C++20. In the past, it has been provided by compilers.

A similar technique can be used to do bgj bucketing.
BGJ15 Bucketing

SimHash-like bucketing bgj15:

- Sample a random vector $r$ and let the vectors in db with similar SimHash to pass through the filter into the bucket.
- Repeat the filtering progress (with different SimHash setting) to get a even smaller bucket.
- Expecting to acquire $O(N)$ shorter vectors from db in $O(N)$ time, where $N$ is the db size.
Fast Collision Detection

Store a hash \( \text{uid} = H(x) \) for every \( x \in \text{db} \). \( H \) has the property that \( H(x_1) \pm H(x_2) \) is the uid for \( x_1 \pm x_2 \).
Fast Lift Rejection - Dual Hash

[DSvW21] has introduced a new tech for fast rejecting vectors that non-likely to be worthy lifted called dual hash.

For context $\kappa, \ell, r$, let $\mathcal{L} = \mathcal{L}[\ell, r]$. To lift $k = \ell - \kappa$ dims, we choose a full row-rank matrix $D \in \mathbb{R}^{h \times k}$ for $h \geq k$, with rows in $\mathcal{L}^*$ and define

$$\mathcal{H}_D : t \mapsto Dt$$

It can be proved that

$$\text{dist}(\mathcal{H}_D(t), \mathbb{Z}^h) \leq \sigma_1(D) \cdot \text{dist}(t, \mathcal{L})$$

where $\sigma_1(D)$ is the largest singular value of $D$. Dual Hash tech says we only lift such vectors that $\mathcal{H}_D(t) \leq H$ for some constant $H$. 
Fast Lift Rejection - Dual Hash

The complexity of lifting is $O(k^2)$, the complexity of computing dual hash is $O(d \cdot k)$, which is larger than that of lifting is $d \geq k$.

We overcome this by precompute “dual hash helper” $Dt$ for every $t$. Due to its homomorphism property, the complexity can be reduce to $O(h)$ for each pair.
G6K Implementation
Stateful Machine in G6K

G6K treat the sieving algorithm as a stateful machine, the full sieving algorithm is done by a series of instructions.

In G6K implementation, it stores the following states

- **B** the lattice basis, $B^\circ$ the orthonormalized basis.
- $\kappa, \ell, r$ s.t. $0 \leq \kappa \leq \ell \leq r \leq d$, where $[\ell, r]$ is the sieving context, $[\kappa, r]$ is the lifting context.
- $\text{db}$ the sieving database of vectors in $L_{[\ell,r]}$
- $c_{\kappa}, \ldots, c_{\ell}$ the lifting candidates s.t. $c_i \in L_{[i, r]}$ or $c_i = \bot$.

However, $B_{[\ell,r]}$ is stored implicitly due to its necessity in computation.
Database Structure

Database is stored by database db and compressed database cdb. In db entry, we keep the following for a vector e:

- e.x: the vector coordinate in basis $B_{[\ell,r]}$ ($v$ on next page);
- e.yr: the vector coordinate under normalized basis $B^\circ_{[\ell,r]}$ ($v^\circ$ on next page);
- e.cv: the SimHash of e;
- e.uid: the hash of e;
- e.len: the length of e;
- e.dual-helper: the information for computing dual hash;
- e.oft-helper: information for computing lifting;

cdb is smaller and always being sorted. And every entry in cdb has an pointer to its pointed entry in db.
**Instructions**

The sieving context can be changed by following instructions.

- **Extend Right ER:** $\mathcal{L}_{[\ell, r]} \rightarrow \mathcal{L}_{[\ell, r+1]}$

  $$(v_\ell, \ldots, v_r) \mapsto (v_0, \ldots, v_\ell, 0)$$
  $$(v_\ell^\circ, \ldots, v_r^\circ) \mapsto (v_\ell^\circ, \ldots, v_r^\circ, 0)$$

- **Shrink Left SL:** $\mathcal{L}_{[\ell, r]} \rightarrow \mathcal{L}_{[\ell+1, r]}$

  $$(v_\ell, \ldots, v_r) \mapsto (v_{\ell+1}, \ldots, v_r)$$
  $$(v_\ell^\circ, \ldots, v_r^\circ) \mapsto (v_{\ell+1}^\circ, \ldots, v_r^\circ)$$

- **Extend Left EL:** $\mathcal{L}_{[\ell, r]} \rightarrow \mathcal{L}_{[\ell-1, r]}$

  $$(v_\ell, \ldots, v_r) \mapsto (-\lfloor c \rfloor, v_\ell, \ldots, v_r)$$
  $$(v_\ell^\circ, \ldots, v_r^\circ) \mapsto ((c - \lfloor c \rfloor) \cdot \left| b_{\ell-1}^* \right|, v_\ell^\circ, \ldots, v_r^\circ)$$
Instructions

Their are some other instructions:

- **Init**$_B$: Initialize the machine with basis $B$.
- **Reset**$_{\kappa,\ell,r}$: Empty the database, set the context to $(\kappa, \ell, r)$.
- **S**: Sieve the database, lift the results in $L_{[\kappa, r]}$. If we enable "on-the-fly lift", then all encountered vector is lifted to $L_{[\kappa, r]}$.
- **I$_i$, I**: Insert candidate at position $i$, or insert to the position with highest scores. The sieving context goes to $[\ell + 1, r]$.
- **Resize**$_N$: Resize the database size to $N$, sample new vectors or remove longest to adapt new size.
Operation Examples

The plain sieve:

\[
\text{Reset}_{[0,0,n-1]}, S, I_0
\]

The progressive sieve:

\[
\text{Reset}_{[0,0,0]}, (ER, S)^n, I_0
\]

Partial HKZ reduction with \( f \) dimensions for free (from left to right):

\[
\text{Reset}_{0,f,f}, (ER, S)^{n-f}, I_0, I_1, \ldots, I_{n-f-1}
\]

Partial HKZ reduction with \( f \) dimensions for free (from right to left):

\[
\text{Reset}_{0,d,d}, (EL, S)^{n-f}, I_0, I_1, \ldots, I_{n-f-1}
\]
Pump Operation

\( \text{Pump}_{\kappa, f, \beta, s} \) sieves for dim \( \beta \), and use \( f \) dim for free, extend from right to left, and always insert to the best position, \( s \) is the switch of down-sieving:

\[
\text{Pump}_{\kappa, f, \beta, s} : \text{Reset}_{\kappa, \kappa + \beta, \kappa + \beta, (EL, S)^{\beta - f}, (I, S^s)^{\beta - f}}
\]

- In exact-SVP, it is suggested to turn off \( s \), i.e., \( s = 1 \).
- In GPU implementation that solves 1.05-SVP, \( s = 1 \).
In practice, we don’t know accurate dim for free (especially when enable on-the-fly lift), to obtain as much dim4free as possible, we test by Workout:

$$\text{Workout}_{\kappa,\beta,f,f^+,s} : \text{Pump}_{\kappa,\beta-f^+,\beta,s}, \text{Pump}_{\kappa,\beta-2f^+,\beta,s}, \text{Pump}_{\kappa,\beta-3f^+,\beta,s}, \ldots, \text{Pump}_{\kappa,\beta-f,\beta,s}$$

where \( f \) is the minimal dim4free. Notice that after every run of Pump, we obtain better basis.
Due to the memory bandwidth of GPU, we no longer insert the obtained vector back to database while sieving. The GPU sieving is separated into 3 phases:

Fig. 2. High level diagram of the implemented Sieving process.
GPU Sieving

Compared to CPU, GPU siever has following characteristics:

- Only $x$ representation is sent to GPU, other information (e.g., $yr$ representation, length) is computed on the fly.
- Inner products of pairs $y_1, \cdots, y_s$ are computed together by $Y^tY$ s.t. $Y = [y_1, \cdots, y_s]$. This can be done very quickly by Tensor cores.
- It seems SimHash is no longer used here.
GPU Sieving

(T)D4F = target/actual dimensions for free
MSD = actual maximum sieving dimension

FLOP = # bucketing + reduction core floating point operations

<table>
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<th>dim</th>
<th>TD4F</th>
<th>D4F</th>
<th>MSD</th>
<th>Norm</th>
<th>Norm/GH</th>
<th>FLOP</th>
<th>Walltime</th>
<th>Mem GiB</th>
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Machine specification:
- 2\times Intel Xeon Gold 6248 (20C/40T @ 2.5-3.9GHz)
- 4\times Gigabyte RTX 2080 TI (4352C @ 1.5-1.8GHz)
- 1.5 TiB RAM (2666 MHz)

Average load: 40 CPU threads @ 93%, 4 GPUs @ 79%/1530MHz/242Watt

Table 1. Darmstadt Lattice 1.05-approxSVP Challenge results
Section 6

Future Plan
First Step

The G6K-GPU seems to be non-well-formed.

1. Rewrite the code with C++ and CUDA in a cmake project, based on our own modified well-formed C++ lattice algorithm library fplll.

2. Follow newest theoretical optimization.

3. Experiment with our own optimization with our own program.

However, the GPU siever seems to be well formed, we might test ideals merely involves siever by modifying the GPU siever.
1. Tuple (> 3) sieve to save memory? For tuples \( V = [v_1, \cdots, v_t] \), we check random \( v_a = Va \) and \( v_b = Vb \) with \( a, b \) sampled uniformly from \( \{-1, 0, 1\}^n \)?

2. Can we enable SimHash in GPU setting? What about group by SimHash? Or can we bucket buckets into sub-buckets according to SimHash?
Section 7

Bibliography
Bibliography I


Bibliography II


Bibliography III
