Fundamentals of Lattice-Based Cryptography

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2nd Crypto Innovation School
Shanghai, China
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Talk Outline

1. Lattices and hard problems
2. The SIS and LWE problems; basic applications
3. Using rings for efficiency

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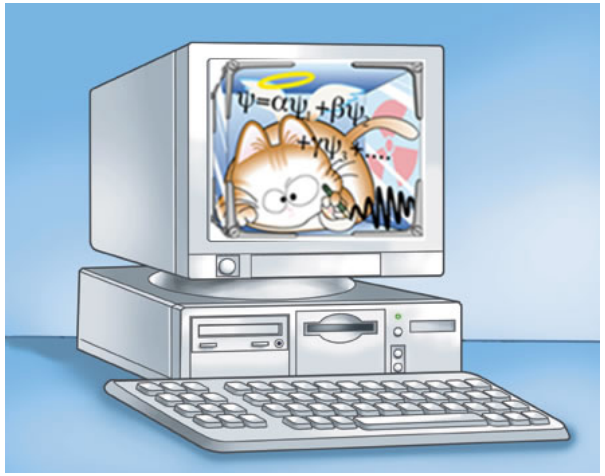
$\longrightarrow$



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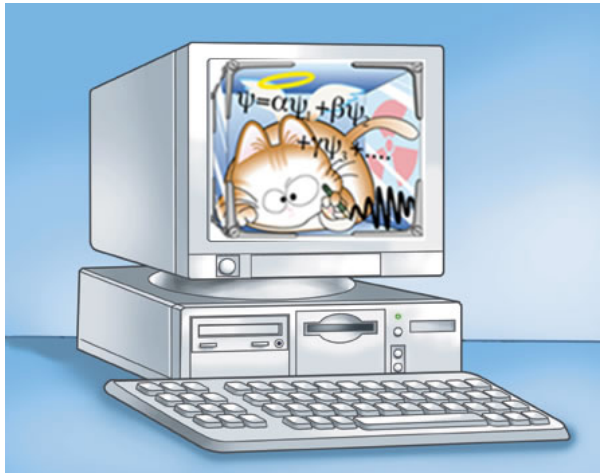

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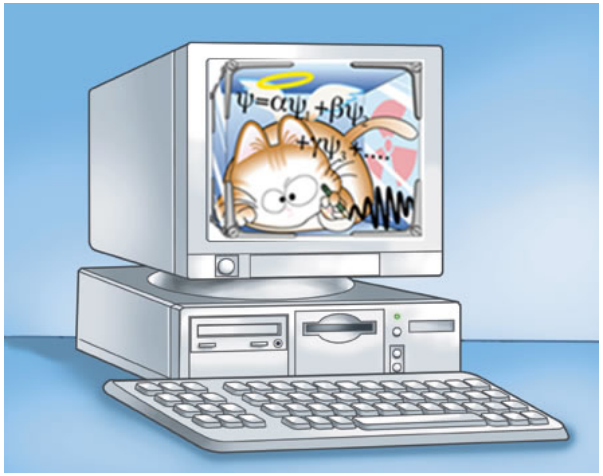
$$g, y = g^X \in G \longrightarrow$$



$$\longrightarrow X$$

Lattice-Based Cryptography

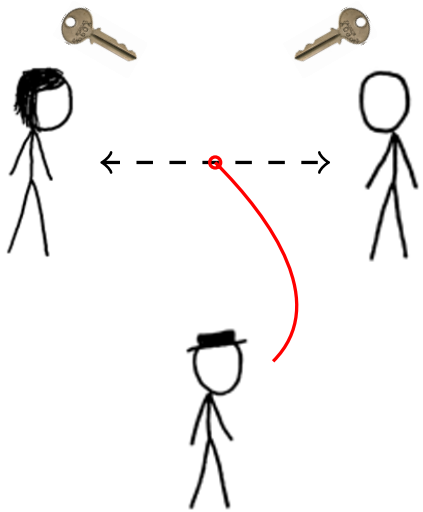
$y = g^x \bmod p$

$N = p \cdot q$

$m^e \bmod N$

$e(g^a, g^b)$

$\Longrightarrow$



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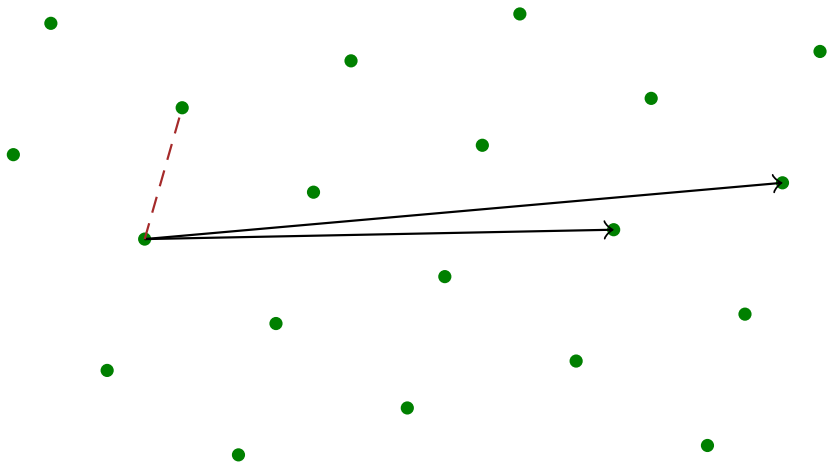

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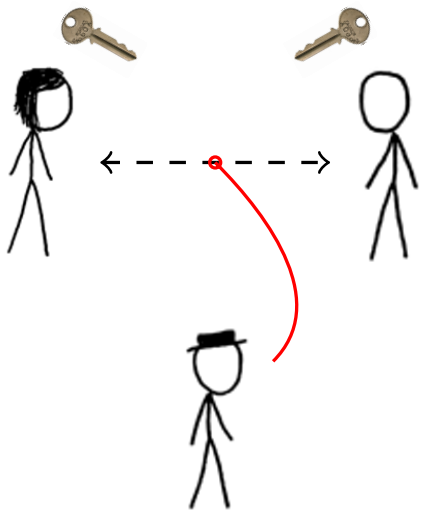

(Images courtesy xkcd.org)

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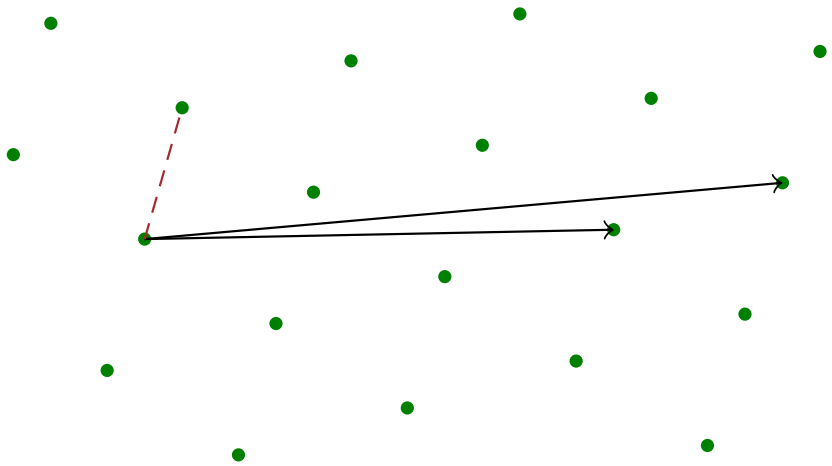

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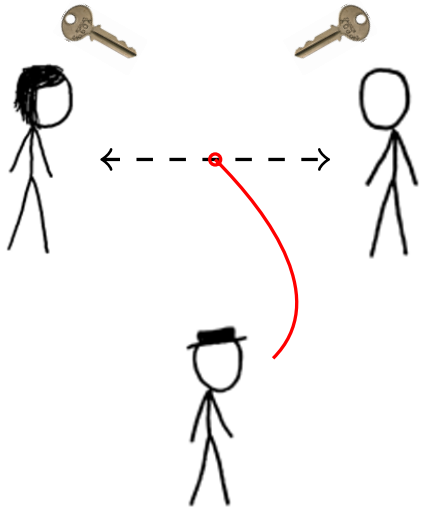

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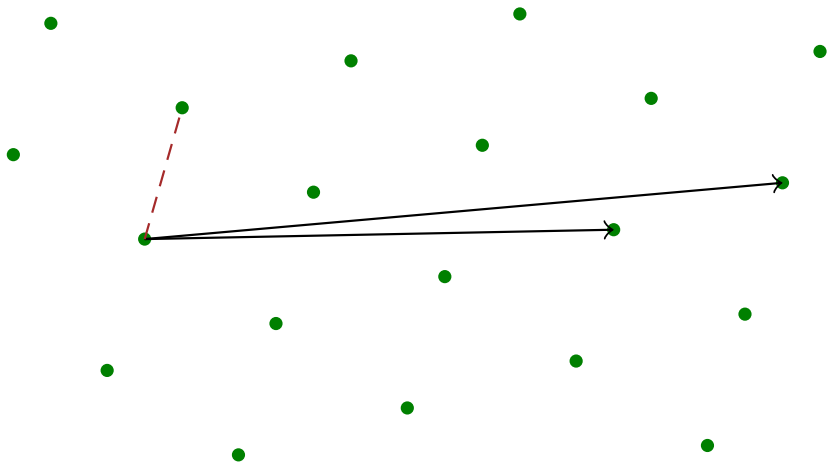

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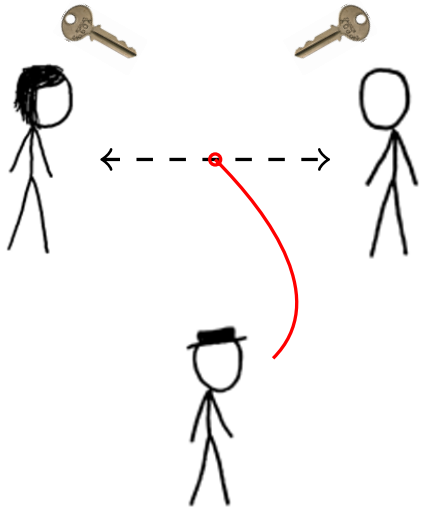

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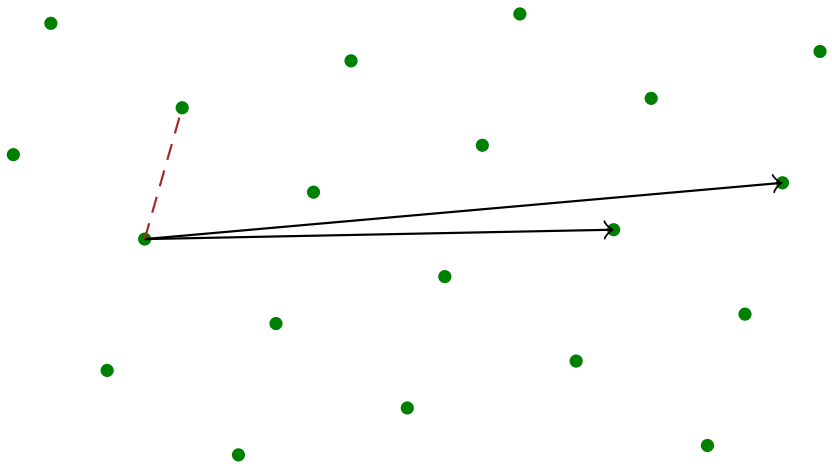

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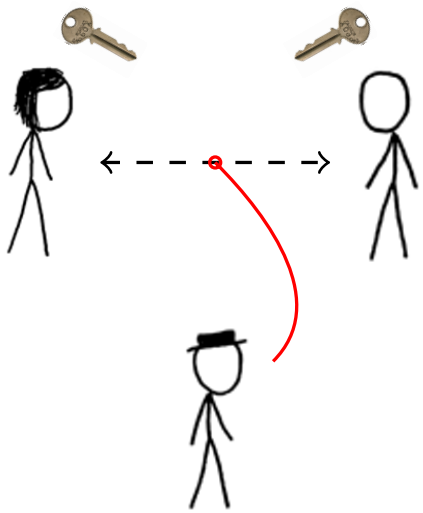

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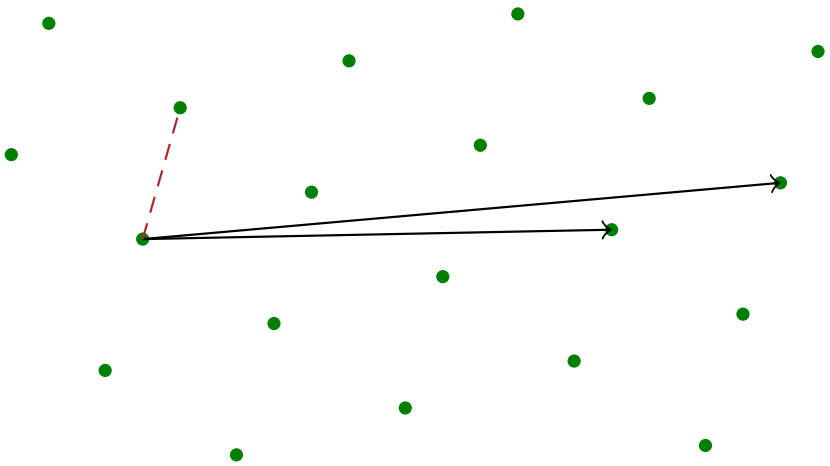

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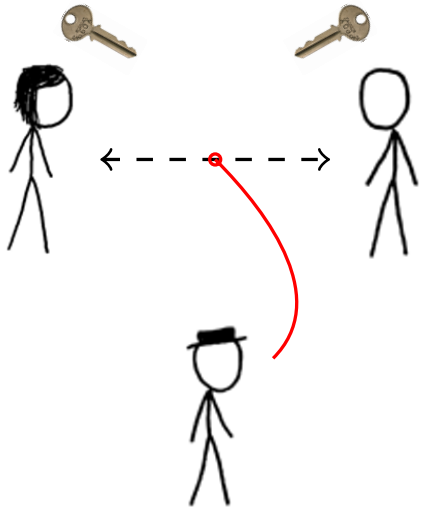

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- Security from worst-case assumptions [Ajtai96,...]

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Part 1:

Lattices and Hard Problems

Lattices

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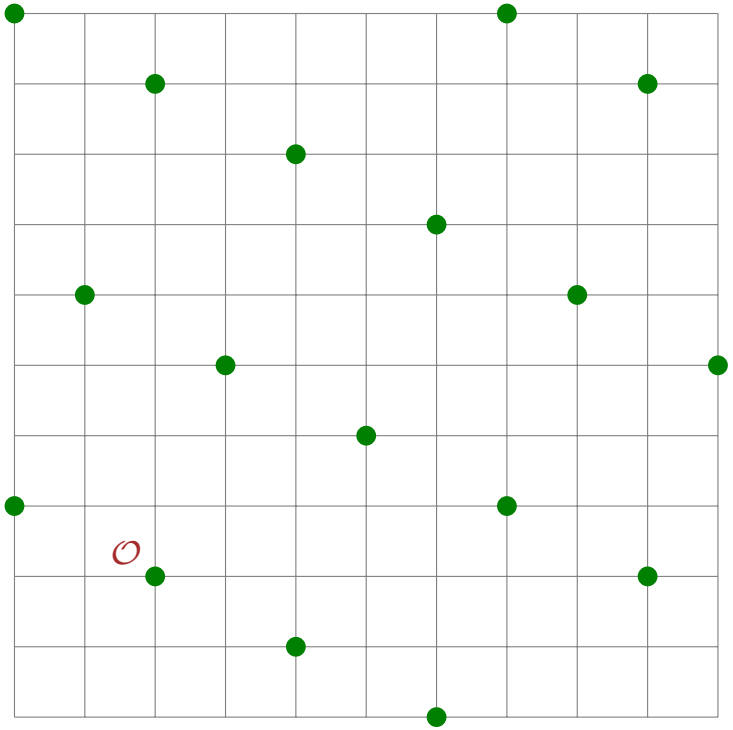

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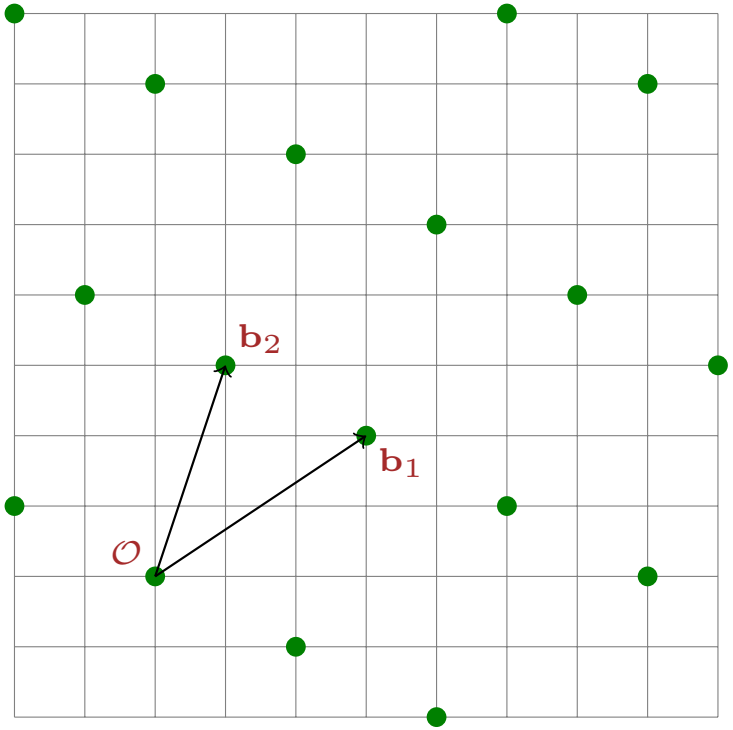

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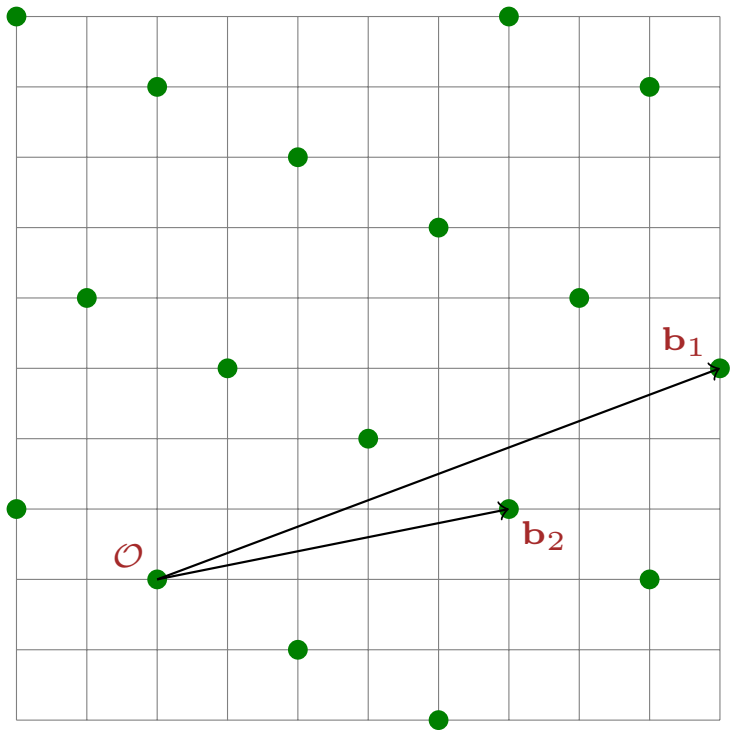

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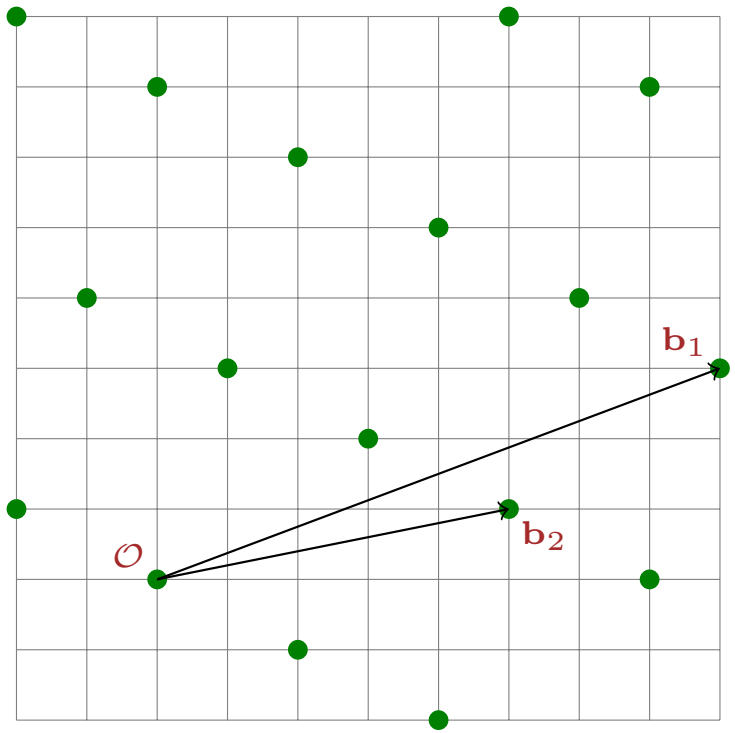

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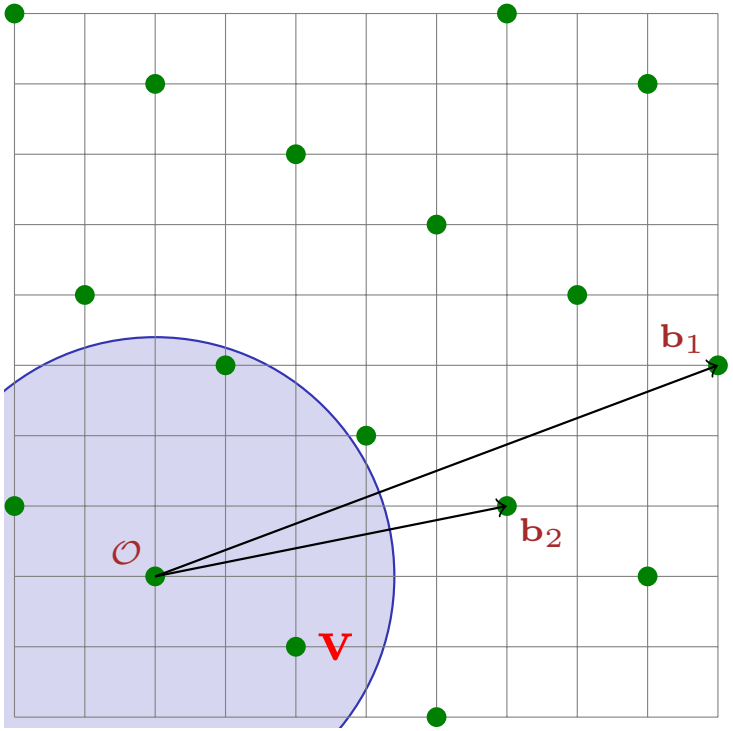

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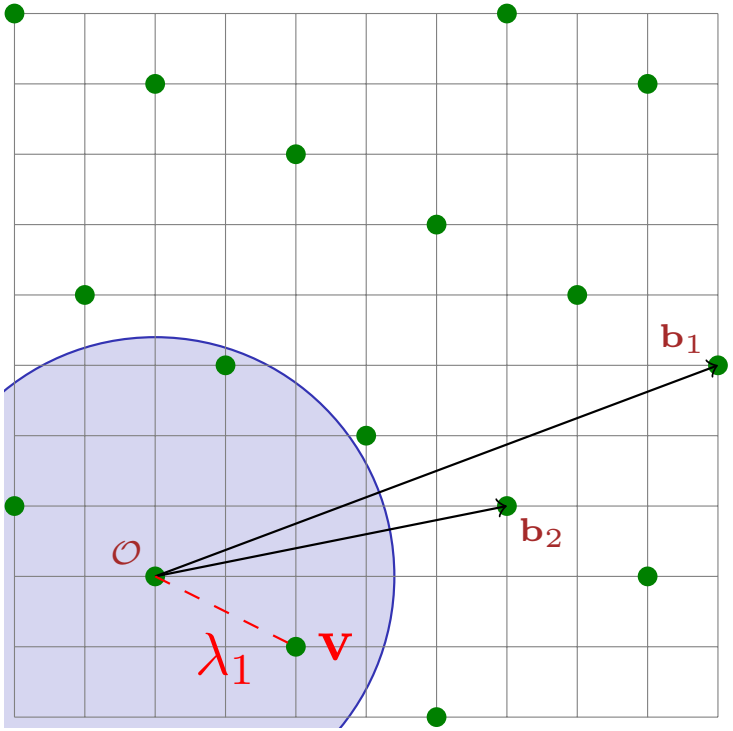

Conjectured Hard Problems

- Find 'relatively short' (nonzero) lattice vector(s): SVP_γ, SIVP_γ
- Estimate geometric quantities of the lattice: minimum distance λ_1, successive minima λ_i, covering radius μ, ...

Complexity (for the Worst Case)

GapSVP$_\gamma$

- Given (a basis of) an m-dim lattice $\mathcal{L}$ and some $d > 0$, distinguish

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- Similar status for other problems like SIVP$_\gamma$, ...

Part 2:

SIS/LWE and Basic Applications

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- **SIS**: given many uniform $\mathbf{a}_i$, find nontrivial $z_1, \ldots, z_m \in \{0, \pm 1\}$ s.t.

$$z_1 \cdot \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + z_2 \cdot \begin{pmatrix} | \\ \mathbf{a}_2 \\ | \end{pmatrix} + \cdots + z_m \cdot \begin{pmatrix} | \\ \mathbf{a}_m \\ | \end{pmatrix} = \begin{pmatrix} | \\ 0 \\ | \end{pmatrix} \in \mathbb{Z}_q^n$$

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  ...yields a short (nonzero) solution $\mathbf{z} = \mathbf{x} - \mathbf{x}' \in \{0, \pm 1\}^m$.

Cool! (but what does this have to do with lattices?)

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- Matrix $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_m) \in \mathbb{Z}_q^{n \times m}$:

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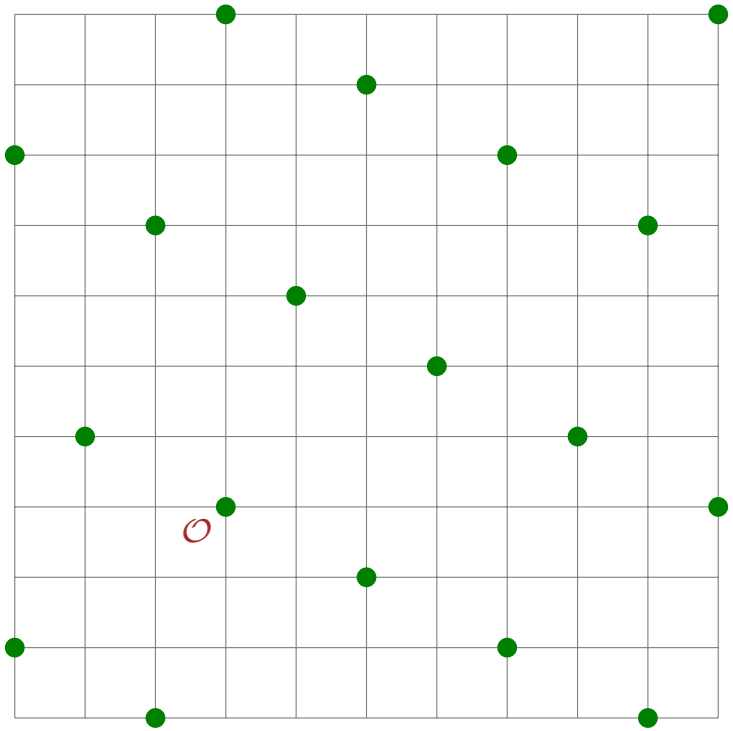

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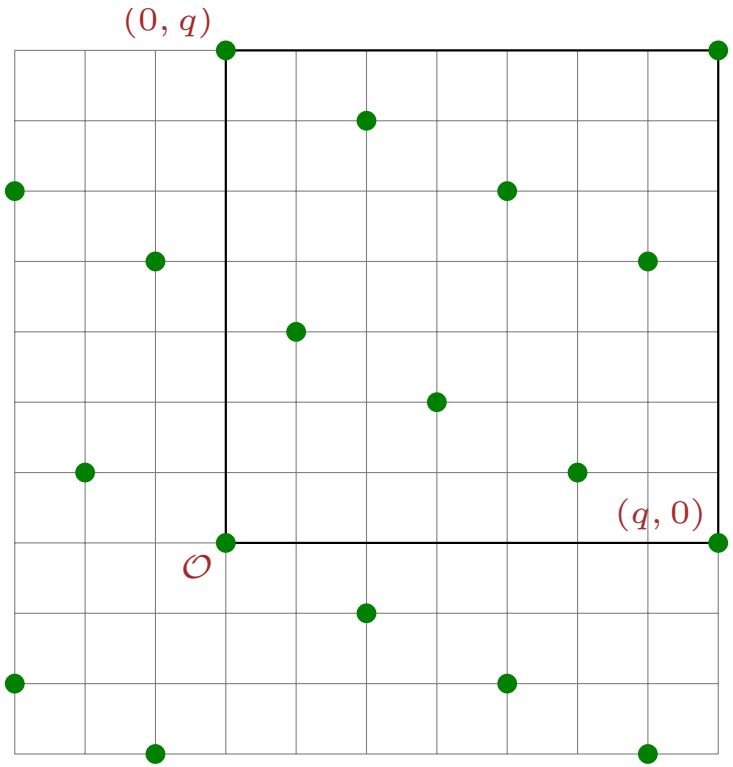

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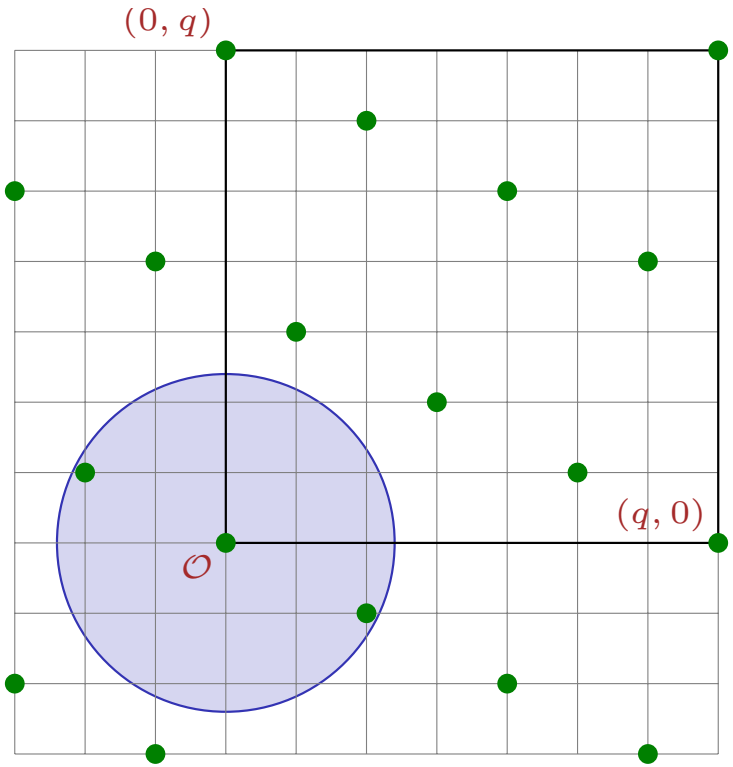

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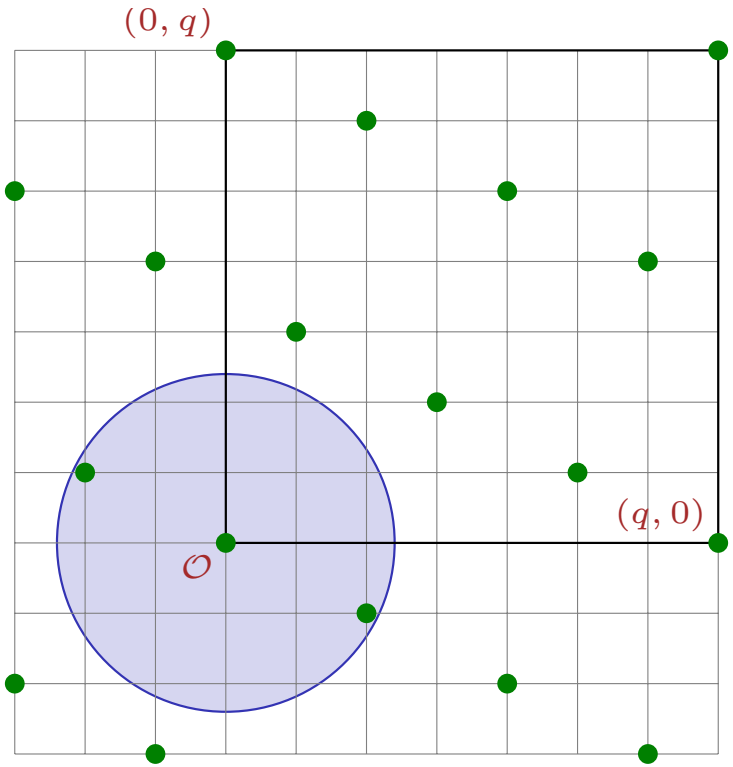

Worst-Case/Average-Case Connection [Ajtai96,...]

Finding 'short' ($\|\mathbf{z}\| \leq \beta \ll q$) nonzero $\mathbf{z} \in \mathcal{L}^{\perp}(\mathbf{A})$

(for uniformly random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$)

$$\Downarrow$$

solving GapSVP$_{\beta\sqrt{n}}$ and SIVP$_{\beta\sqrt{n}}$ on any n-dim lattice

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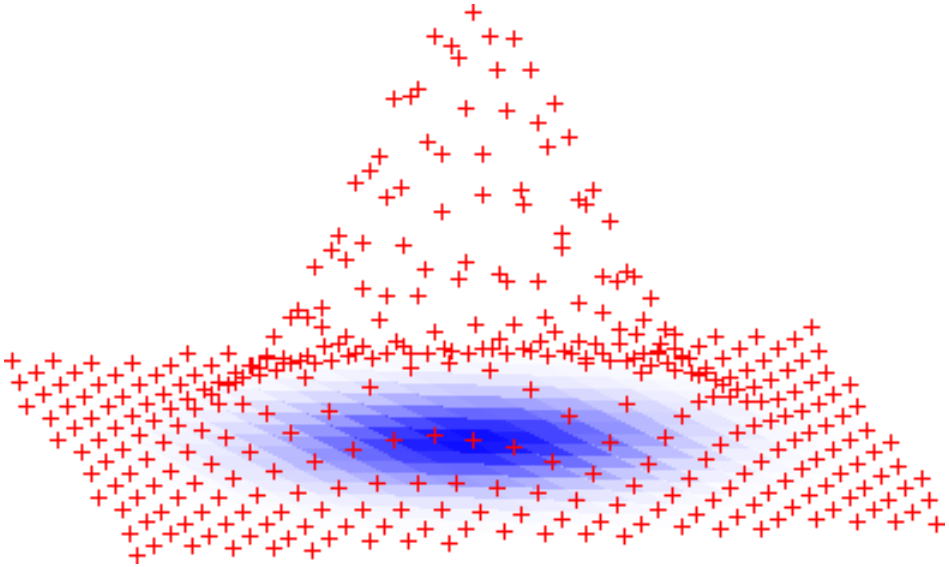

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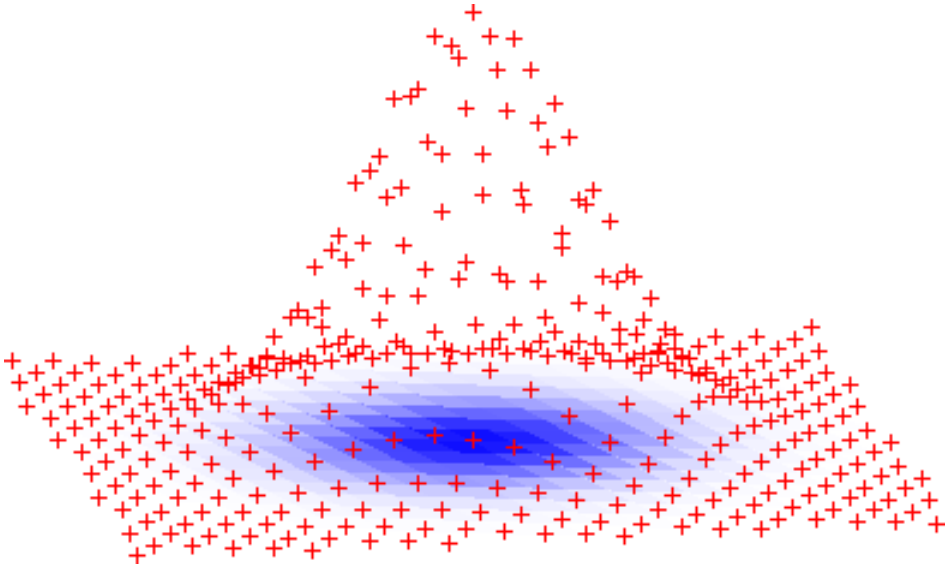

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Application: Digital Signatures [GentryPeikertVaikuntanathan'08]

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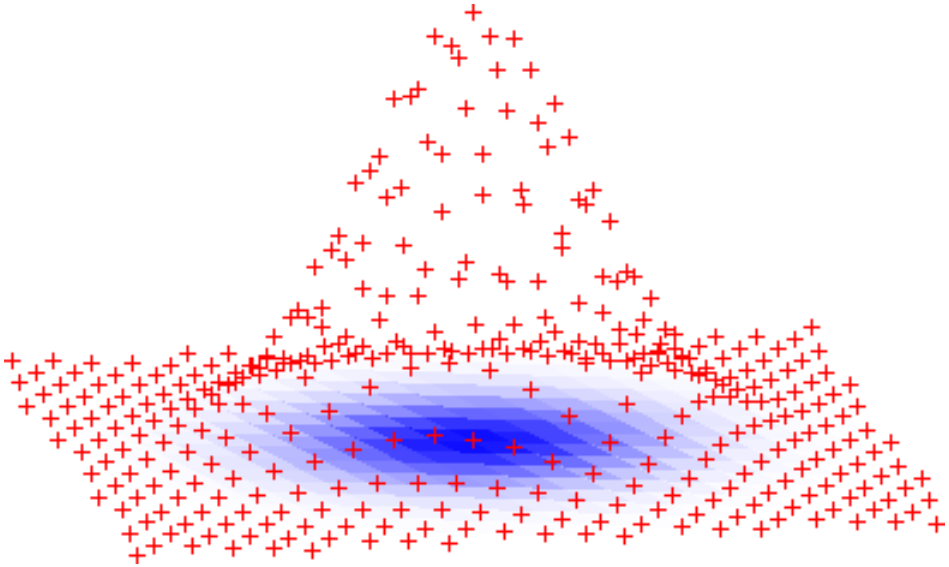

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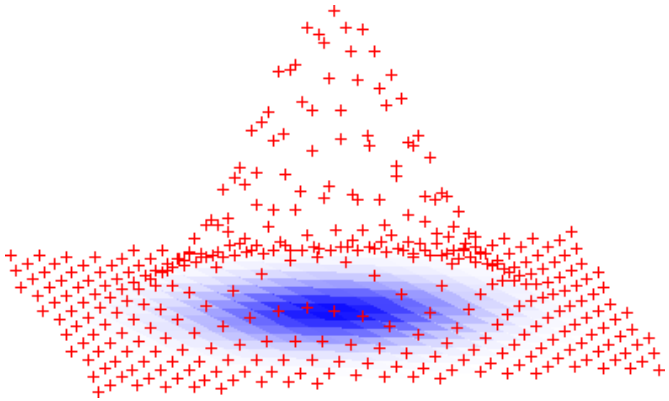

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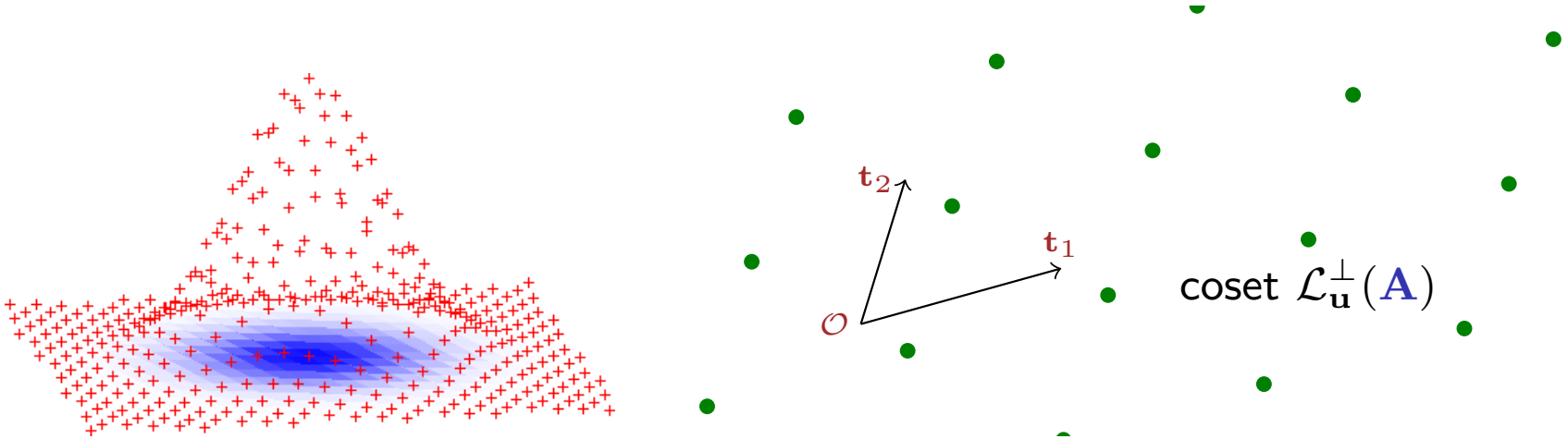

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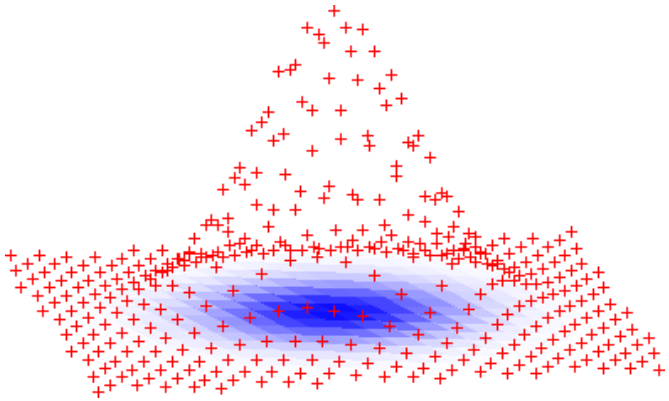

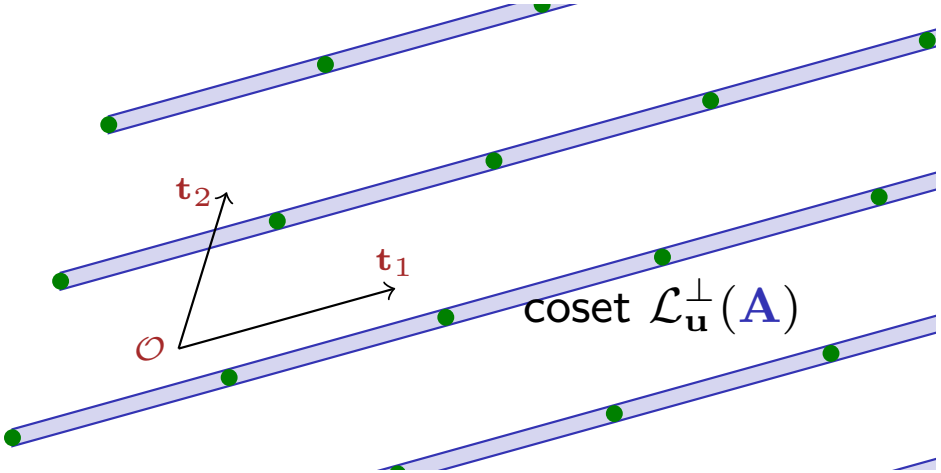

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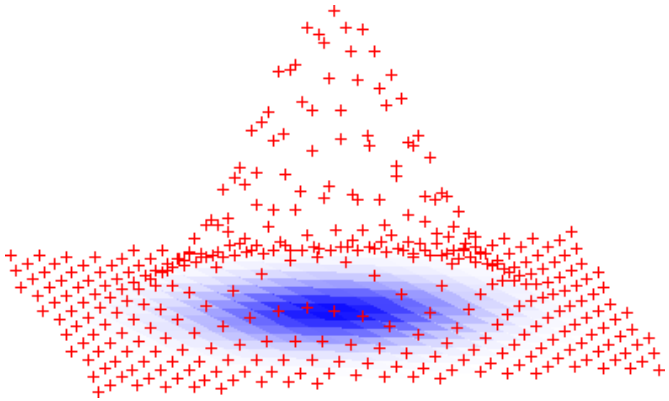

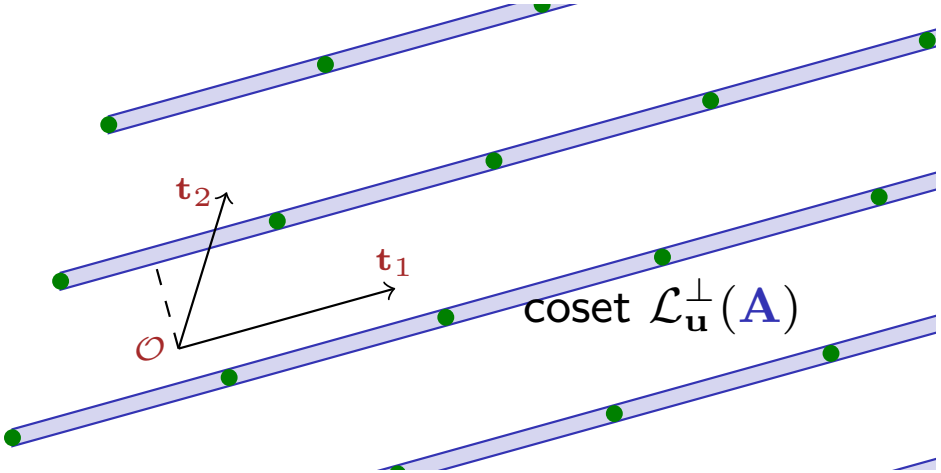

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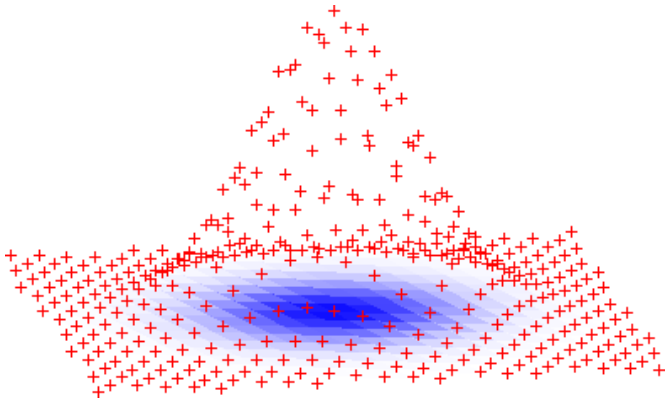

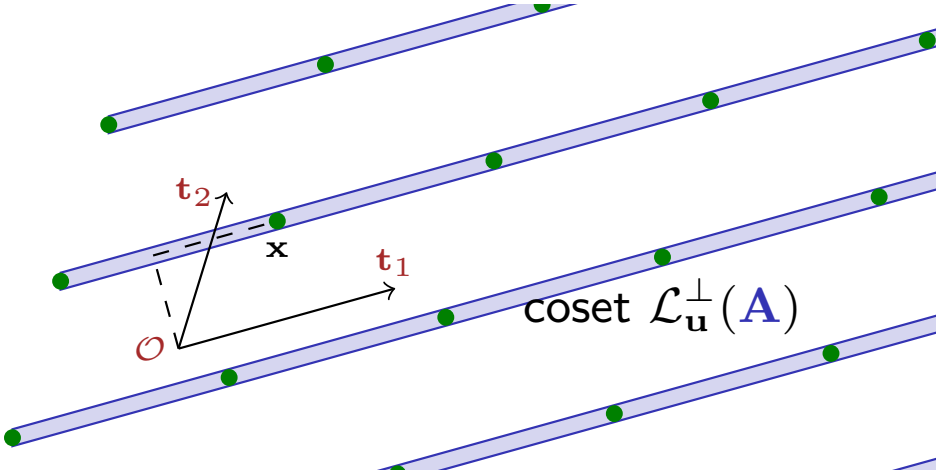

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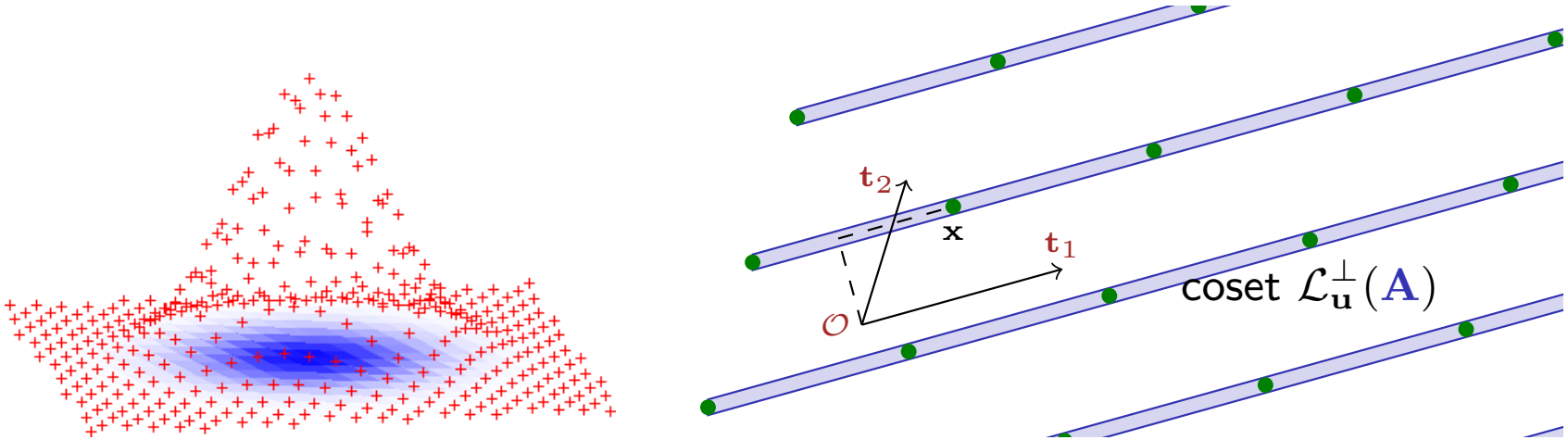

- Proof idea: $D_{\mathcal{L}_{\mathbf{u}}^{\perp}}(\text{plane})$ depends (essentially) only on $\mathrm{dist}(\mathcal{O}, \text{plane})$; not affected by shift within plane. So rounding with that probability produces that distribution.

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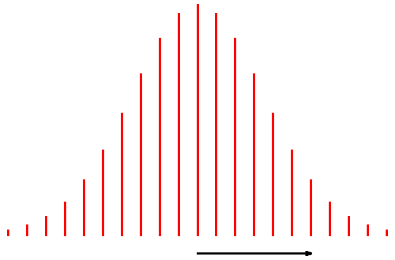

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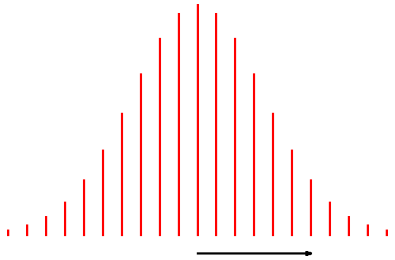

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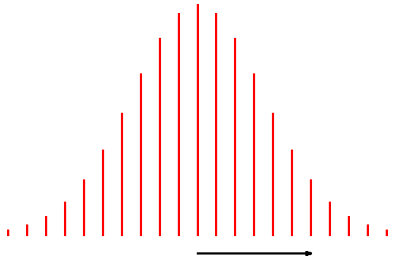

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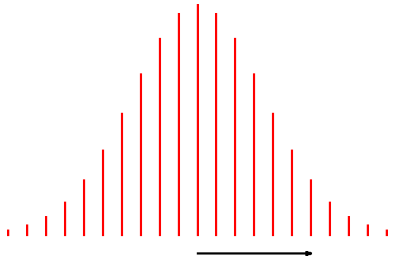

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(n/α)-approx *worst case* GapSVP, SIVP $\leq$ search-LWE $\leq$ decision-LWE $\leq$ crypto

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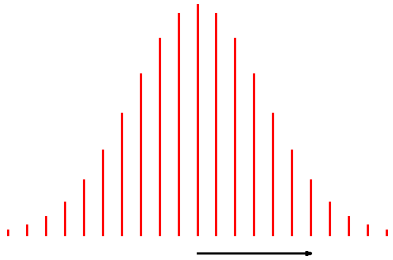

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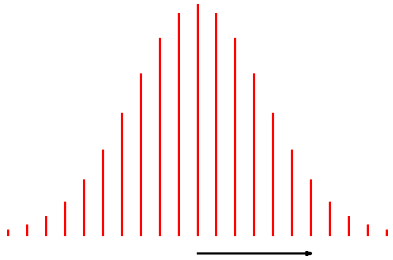

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- !!! Fully Homomorphic Encryption
- !!! Attribute-Based/Predicate Encryption for arbitrary policies

and much, much more...

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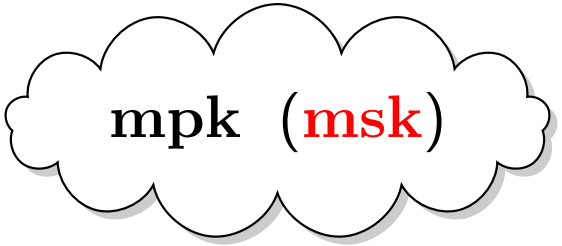

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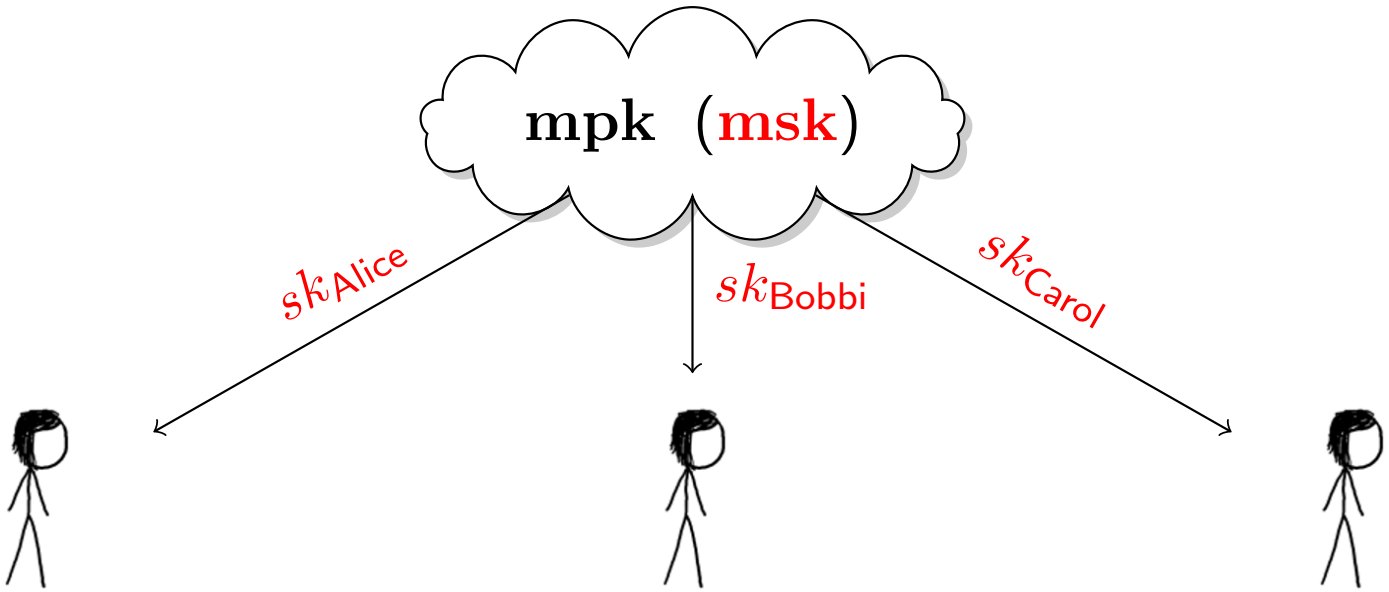

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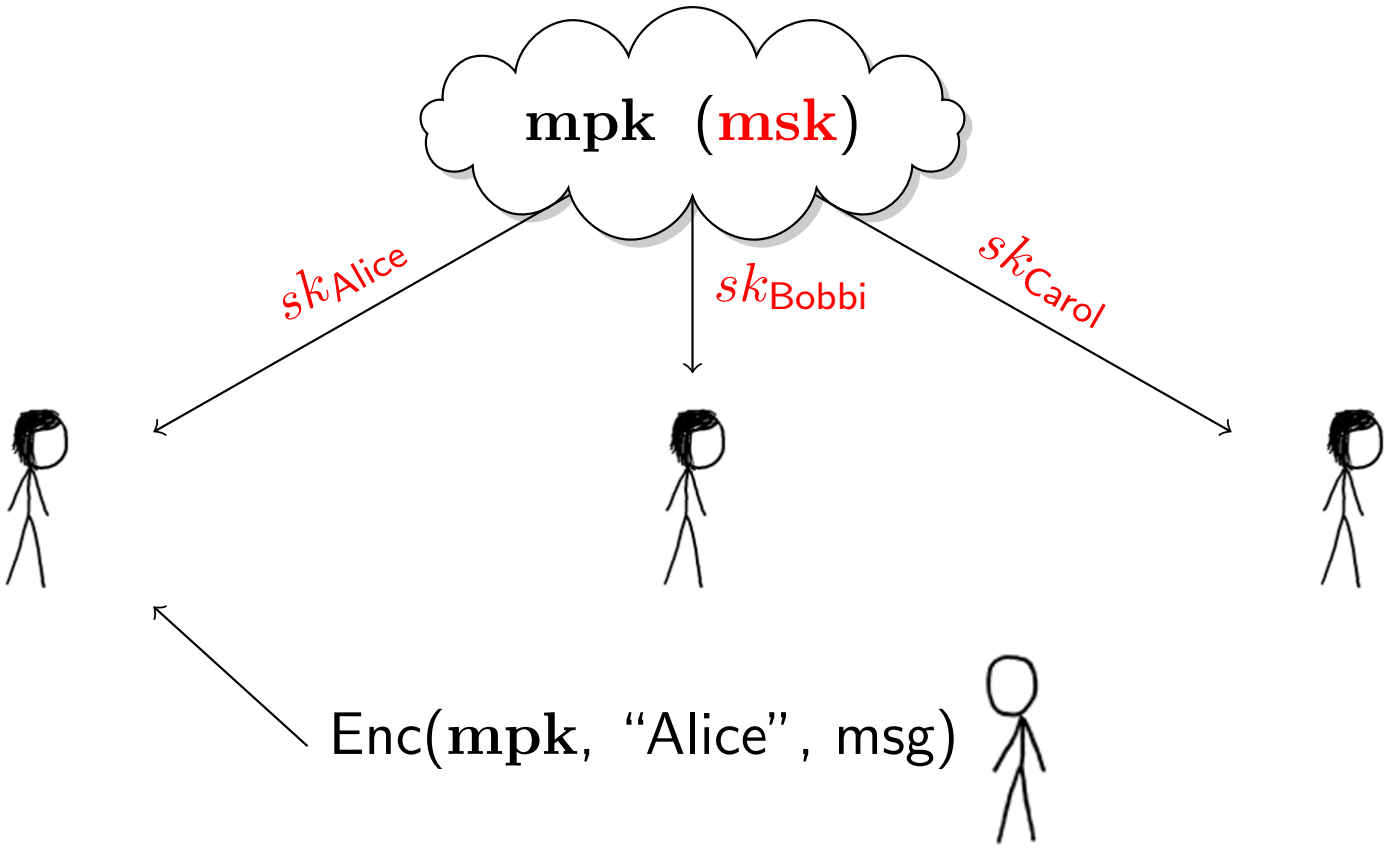

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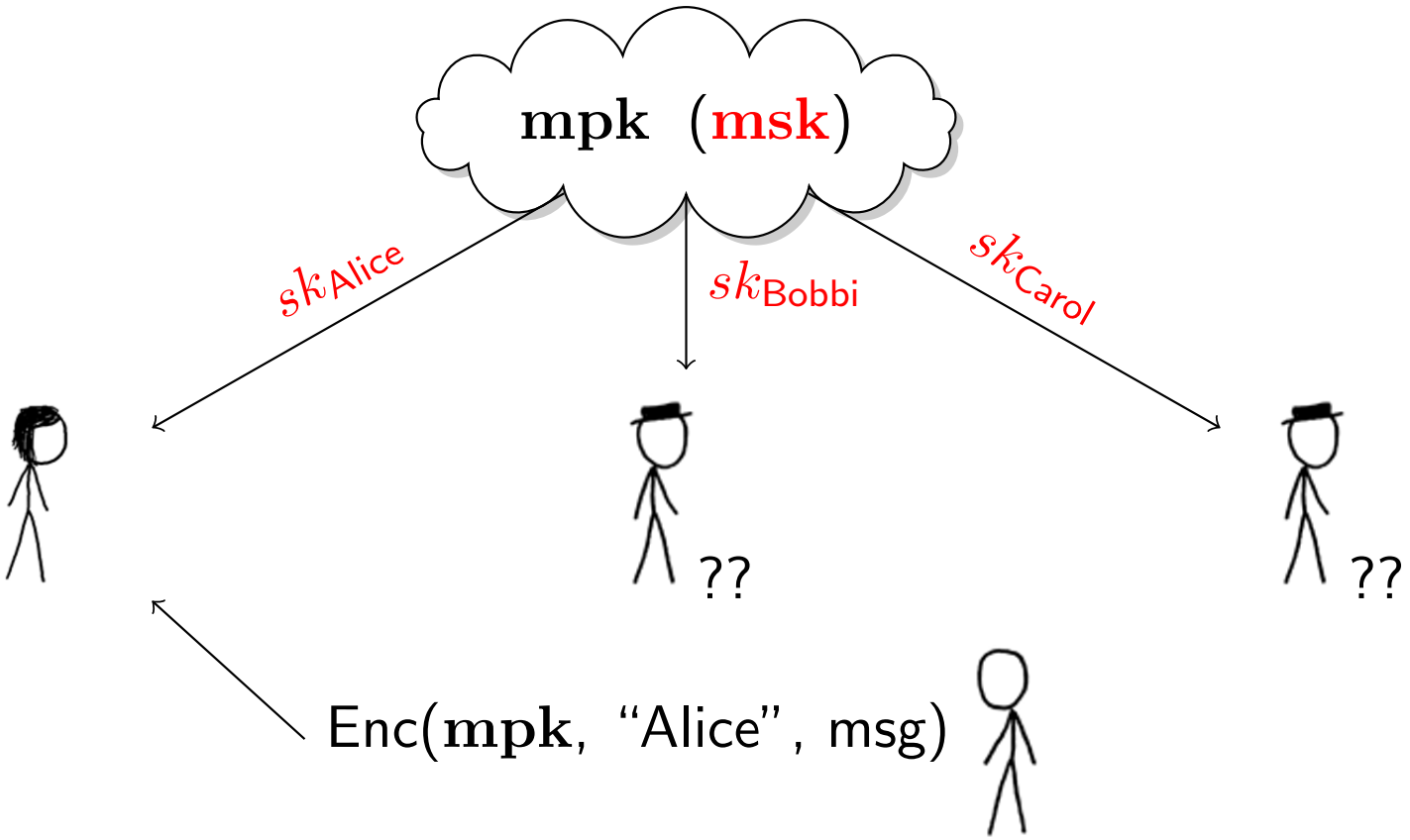

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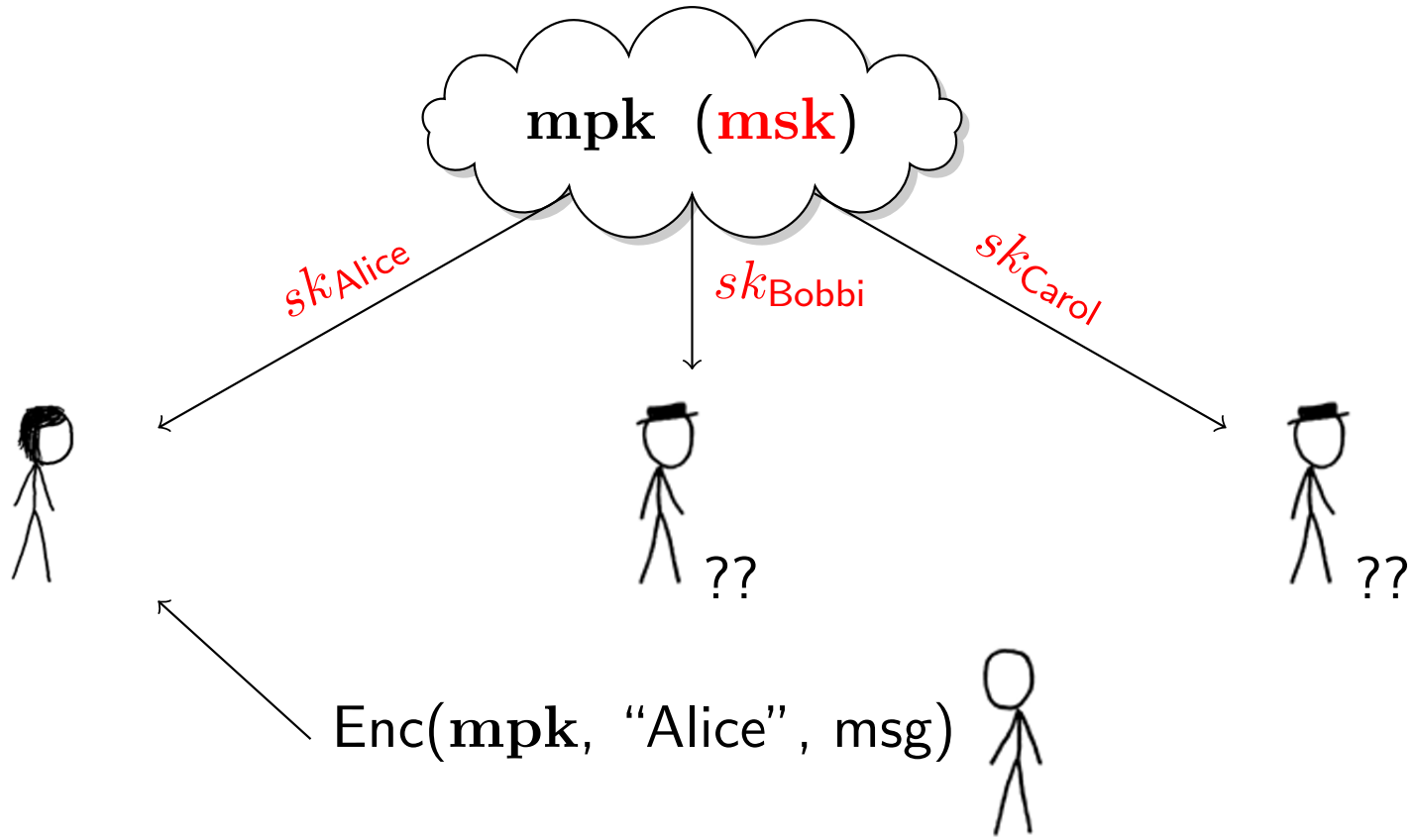

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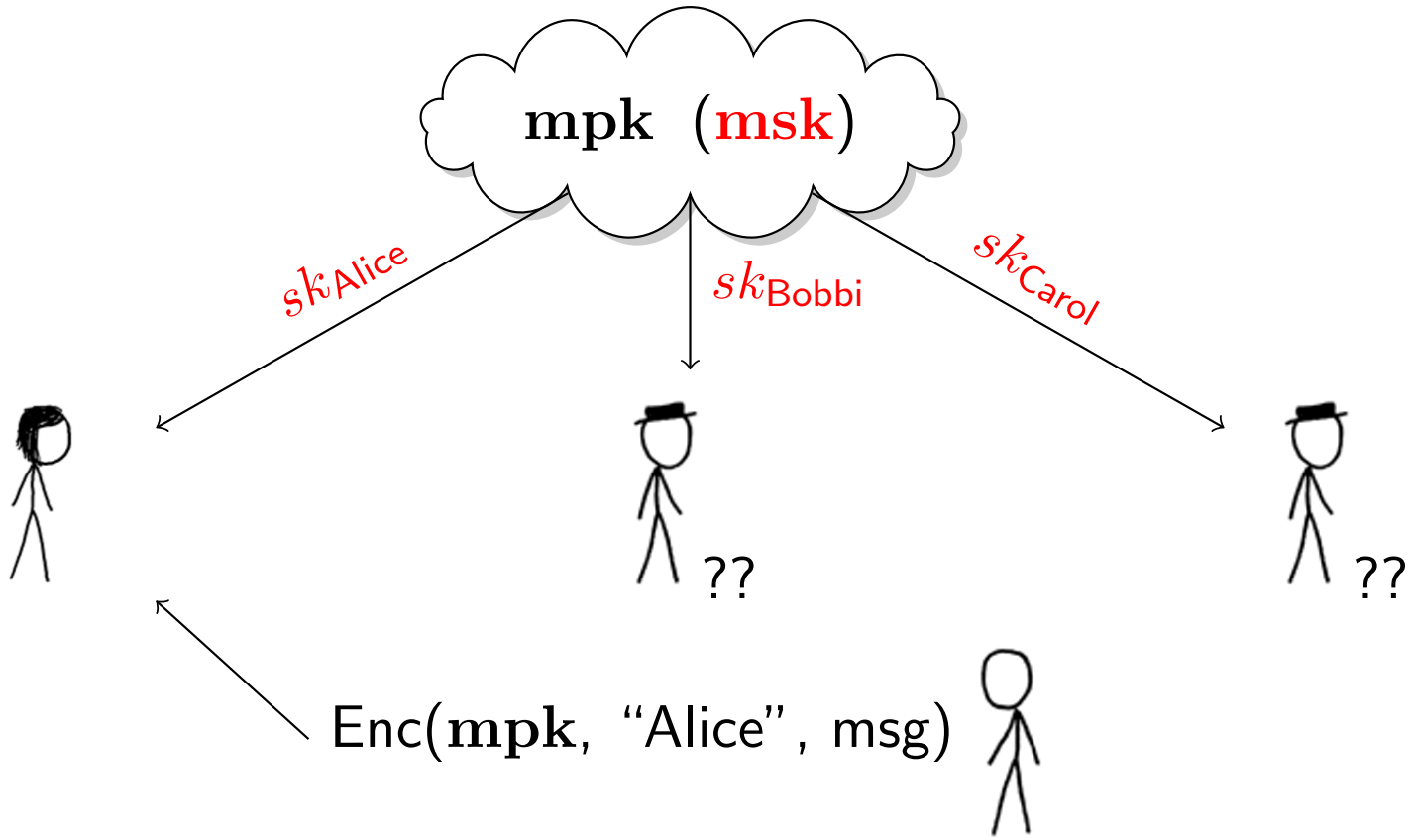

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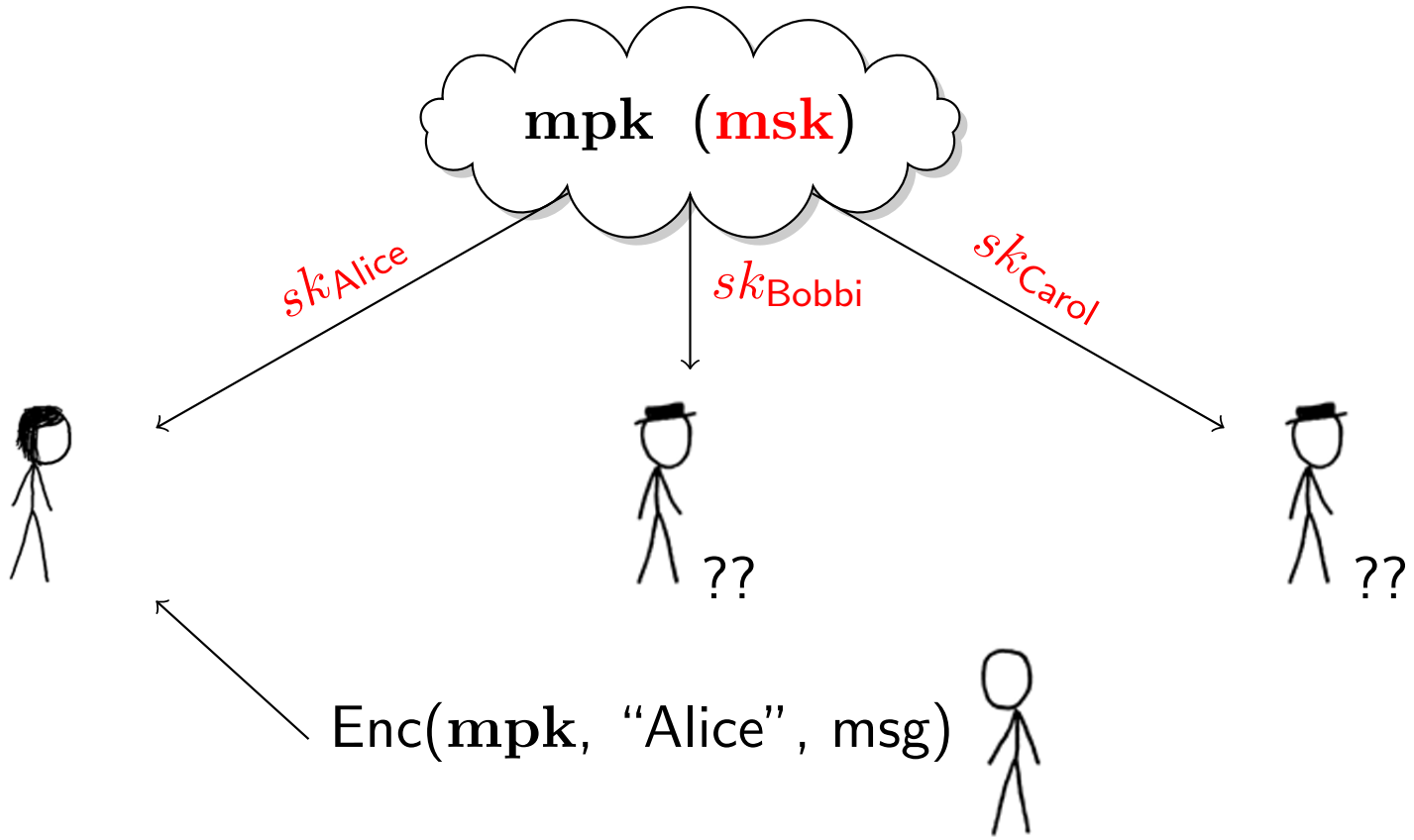

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3. [GPV'08]: based on lattices!

IBE from LWE

Gaussian $\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{u}$

$$mpk = \mathbf{A}$$
$$msk = \text{trapdoor } \mathbf{T}$$

$$\mathbf{u} = H(\text{“Alice”})$$

(‘identity’ public key)

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Part 3:

Rings for Better Efficiency

SIS/LWE are (Sort Of) Efficient

$$\left(\cdots \mathbf{a}_i \cdots\right) \begin{pmatrix} \vdots \\ \mathbf{s} \\ \vdots \end{pmatrix} + e_i \;=\; b_i \in \mathbb{Z}_q$$

- Getting one pseudorandom scalar $b_i \in \mathbb{Z}_q$ requires an n-dim inner product (mod q)

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- Same ring structures used in NTRU cryptosystem [HPS'98],

  compact one-way / CR hash functions [Mic'02,PR'06,LM'06,...]

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- **Decision**: <u>distinguish</u> $(a_i\ ,\ b_i)$ from uniform $(a_i\ ,\ b_i) \in R_q \times R_q$

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Initial Reductions [LyubashevskyPeikertRegev'10]

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Constructions

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Thanks!